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# Microgravity Vibration Isolation: Optimal Preview and Feedback Control

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paper offers five different models in response, followed by development of a useful general expression for the optimal control.

## MATHEMATICAL MODELS

The basic vibration isolation approach that was presented in reference 3 is developed more generally below. It assumes a state-space expression for the system equations of motion, having the form

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} \underline{u} + \underline{f} \quad (1)$$

### Single-Mass Models

Suppose the experiment is modeled as a mass,  $m$ , with position,  $x(t)$ . Let the space station wall have position,  $d(t)$ . If the umbilicals connecting the wall to the experiment can be modeled by a spring and a damper of stiffness,  $k$ , and damping,  $c$ , respectively; the system can be modeled schematically as in figure 1. A magnetic actuator applying a control force  $\alpha i(t)$  in response to a control current  $i(t)$  has been included in the figure.

The system equation of motion is:

$$m\ddot{x} + c(\dot{x} - \dot{d}) + k(x - d) + \alpha i = 0. \quad (2)$$

In state space notation this becomes,

$$\dot{\underline{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{\alpha}{m} \end{Bmatrix} u + \begin{Bmatrix} 0 \\ \frac{k}{m} d(t) + \frac{c}{m} \dot{d}(t) \end{Bmatrix} \quad (3)$$

where:  $x_1 = x$ ,  $x_2 = \dot{x}$ , and  $u = i$ . This is the model which was used in reference 3, i.e., an absolute-state model with a position-plus-velocity disturbance. As shown in reference 3, a quadratic cost function of  $J = 1/2 \int_0^\infty (\underline{x}^T \underline{w}_1 \underline{x} + w_3 u^2) dt$  effectively weights low frequency acceleration disturbances by a factor proportional to  $1/\omega^4$ .

The system could alternatively be represented by a relative-state model with a velocity disturbance, having the form:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{\alpha}{m} \end{Bmatrix} u + \begin{Bmatrix} -1 \\ -\frac{c}{m} \end{Bmatrix} \dot{d} \quad (4)$$

where  $x_1$  and  $x_2$  are now the relative states  $x-d$  and  $\dot{x} - \dot{d}$ , respectively. The same quadratic performance index as before could be used. This second model has the advantage of allowing relative states to be weighted, but it does not permit direct acceleration weighting, since  $\ddot{x}$  is not represented as a state.

A third model corrects this deficiency by adding  $u(t) = i(t)$  as a state  $x_3$  and noting that

$$\ddot{x} = \begin{bmatrix} \frac{-k}{m} & \frac{-c}{m} & \frac{-\alpha}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}. \quad \text{States } x_1 \text{ and } x_2 \text{ are the relative states defined previously. The equation}$$

$$\ddot{x}^2 = \left( \frac{-k}{m} \right)^2 x_1^2 + \left( \frac{-c}{m} \right)^2 x_2^2 + \left( \frac{-\alpha}{m} \right)^2 x_3^2 + \left( \frac{2kc}{m^2} \right) x_1 x_2 + \left( \frac{2c\alpha}{m^2} \right) x_2 x_3 + \left( \frac{2k\alpha}{m^2} \right) x_1 x_3 \quad (5a)$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{k^2}{m^2} & \frac{kc}{m^2} & \frac{k\alpha}{m^2} \\ \frac{kc}{m^2} & \frac{c^2}{m^2} & \frac{c\alpha}{m^2} \\ \frac{k\alpha}{m^2} & \frac{c\alpha}{m^2} & \frac{\alpha^2}{m^2} \end{bmatrix}}_{W_{11}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad (5b)$$

$$= \underline{w}^T W_{11} \underline{x} \quad (5c)$$

allows experiment acceleration to be weighted directly in the performance index, as shown below. The state equations can now be written as

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u + \underline{f} \quad (6a)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-k}{m} & \frac{-c}{m} & \frac{-\alpha}{m} \\ 0 & 0 & 0 \end{bmatrix} \quad (6b)$$

$$\underline{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (6c)$$

$$\mathbf{u} = \frac{di}{dt} \quad (6d)$$

$$\underline{\mathbf{f}} = \begin{bmatrix} 0 \\ -\ddot{d} \\ 0 \end{bmatrix} \quad (6e)$$

If the performance index is

$$J = \frac{1}{2} \int_0^{\infty} (\rho_1 \underline{\mathbf{x}}^T \mathbf{W}_{11} \underline{\mathbf{x}} + \underline{\mathbf{x}}^T \mathbf{W}_{12} \underline{\mathbf{x}} + w_3 u^2) dt, \quad (7)$$

where  $\mathbf{W}_{11}$  is as defined in equation (5), then  $\rho_1$  can be used to weight  $\ddot{x}$ ;  $\mathbf{W}_{12}$  can be used to weight  $x-d$ ,  $\dot{x}-\dot{d}$ , and  $i(t)$ ; and  $w_3$  can be used to weight the slew rate  $u = di/dt$ .

Equation (7) can be rewritten as

$$J = \frac{1}{2} \int_0^{\infty} (\underline{\mathbf{x}}^T \mathbf{W}_1 \underline{\mathbf{x}} + w_3 u^2) dt \quad (8a)$$

where

$$\mathbf{W}_1 = \rho_1 \mathbf{W}_{11} + \mathbf{W}_{12} \quad (8b)$$

and

$$u = \frac{di}{dt} \text{ as in equation (6c).} \quad (8c)$$

Once a satisfactory constant feedforward gain has been determined (by the method to be developed in this paper) the control could be implemented according to the block diagram of figure (2a) or, equivalently, figure (2b). This third model, a relative-state model with direct acceleration weighting and an acceleration disturbance, allows for relative-state weighting, direct acceleration weighting, and slew-rate weighting. It has the disadvantage, however, that it assumes the system parameters to be known well enough for equation (5) to provide a sufficiently accurate estimate of  $\ddot{x}$ . Such an assumption, of course, may not be the case with umbilicals.

### Two-Mass Models

Suppose now that the system is modeled using two masses, with  $m_1$  and  $m_2$  representing the experiment mass and an attached accelerometer mass. Let  $k_1$  and  $c_1$  be the umbilical stiffness and damping, and let  $k_2$  and  $c_2$  be the stiffness and damping of the accelerometer. The model is shown schematically in figure 3.

If the system states are defined as follows,

$$x_0 = \int_0^t [\ddot{x}_1(\tau) - \ddot{x}_2(\tau)] d\tau \quad (9a)$$

$$x_1 = \ddot{x}_1 - \ddot{d} \quad (9b)$$

$$x_2 = \ddot{x}_2 - \ddot{x}_1 \quad (9c)$$

$$x_3 = \dot{x}_1 = \dot{\ddot{x}}_1 - \dot{\ddot{d}} \quad (9d)$$

$$x_4 = \dot{x}_2 = \dot{\ddot{x}}_2 - \dot{\ddot{x}}_1 \quad (9e)$$

then the state equations and performance index can be expressed by equations (10a) and (10b), respectively, where  $u(t) = i(t)$ .

$$\begin{Bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-k_1}{m_1} & \frac{k_2}{m_1} & \frac{-c_1}{m_1} & \frac{c_2}{m_1} & 0 \\ \frac{k_1}{m_1} & \frac{-k_2}{m_2} - \frac{k_2}{m_1} & \frac{c_1}{m_1} & \frac{-c_2}{m_2} - \frac{c_2}{m_1} & 0 \end{bmatrix} \begin{Bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{-\alpha}{m_1} \\ \frac{\alpha}{m_1} \end{Bmatrix} u + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\ddot{d} \\ 0 \end{Bmatrix} \quad (10a)$$

$$J = \frac{1}{2} \int_0^{\infty} (\underline{x}^T W_1 \underline{x} + w_3 u^2) dt \quad (10b)$$

In this formulation  $\ddot{x}$  is approximated closely by the state  $x_2$ , which is available as the accelerometer output voltage. This relative-state model with accelerometer feedback and an acceleration disturbance adds to the complexity of the problem. However, it allows the experiment acceleration to be represented directly as a state. It is necessary only that the accelerometer mass, stiffness, and damping be known with reasonable accuracy.

A system making use of a load cell instead of an accelerometer, as portrayed in figure 4, would lead to very similar state equations. Let  $k_1$  and  $c_1$  describe the umbilical characteristics as before. If a load cell of stiffness  $k_2$  separates the experiment mass  $m_2$  from a small plate of mass  $m_1$  attached to the end of the magnetic actuator, then the state equations can be expressed by equation (11).

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_2}{m_1} & -\frac{c_1}{m_1} & 0 & 0 \\ \frac{k_1}{m_1} & -\frac{k_2}{m_2} - \frac{k_2}{m_1} & \frac{c_1}{m_1} & 0 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\alpha}{m_1} \\ \frac{\alpha}{m_1} \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\ddot{d} \\ 0 \end{pmatrix} \quad (11)$$

Here the states, performance index, and control are defined again by equation (9). This relative-state model with load-cell feedback and an acceleration disturbance also allows the experiment acceleration to be represented directly as a state (viz.,  $x_2$ ). It has the further advantages that  $k_2$  can be known with great accuracy and that the load cell output gives an accurate state measurement even with very low frequencies.

All five of the above models have the form

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} + \underline{f} \quad (12a)$$

and cost function

$$J = \frac{1}{2} \int_0^{\infty} (\underline{x}^T \underline{W}_1 \underline{x} + \underline{u}^T \underline{W}_3 \underline{u}) dt \quad (12b)$$

These equations, along with the initial conditions

$$\underline{x}(0) = \underline{x}_0 \quad (12c)$$

and the terminal conditions

$$\lim_{t \rightarrow \infty} \underline{u}(t) = \underline{0} \quad (12d)$$

$$\lim_{t \rightarrow \infty} \underline{f}(t) = \underline{0} \quad (12c)$$

have already been treated analytically in reference 3. However, an analytical solution is very algebraically intensive. For this reason we seek a general matrix solution that is more amenable to numerical treatment, to accommodate better the higher order systems. This paper will use a differential equations approach to find such a solution. A state transition matrix approach that yields confirmatory results will be the subject of a later paper.

## OPTIMAL CONTROL PROBLEM

The optimal control problem is to find the control  $\underline{u}(t)$  which minimizes the quadratic performance index

$$J = \frac{1}{2} \int_0^{\infty} (\underline{x}^T W_1 \underline{x} + \underline{u}^T W_3 \underline{u}) dt \quad (13)$$

subject to the conditions

$$\underline{x}(0) = \underline{x}_0 \quad (14a)$$

$$\lim_{t \rightarrow \infty} \underline{x}(t) = \underline{0} \quad (14b)$$

$$\lim_{t \rightarrow \infty} \underline{f}(t) = \underline{0} \quad (14c)$$

where  $W_1$  and  $W_3$  are the state- and control-weighting matrices, respectively.

## SOLUTION

The argument of the cost function  $J$  from equation (13) is first augmented by the Lagrange multiplier  $\underline{\lambda}(t)$  times the state equation of motion equation (12a), to yield the equation

$$\bar{J} = \int_0^{\infty} H dt \quad (15a)$$

where the Hamiltonian  $H$  is

$$H = \frac{1}{2} (\underline{x}^T W_1 \underline{x} + \underline{u}^T W_3 \underline{u}) + \underline{\lambda} (\dot{\underline{x}} - A \underline{x} - B \underline{u} - \underline{f}) \quad (15b)$$

It is desired to obtain an optimal solution  $\underline{u} = \underline{u}^*$  which minimizes  $\bar{J}$ .



The first variation of  $\mathcal{J}(\underline{x}, \underline{u}, \underline{\dot{x}})$  is

$$\delta \mathcal{J} = \int_0^\infty \left[ \frac{\partial H}{\partial \underline{x}} \delta \underline{x} + \frac{\partial H}{\partial \underline{u}} \delta \underline{u} + \frac{\partial H}{\partial \underline{\dot{x}}} \delta \underline{\dot{x}} \right] dt$$

The control  $\underline{u}$  which satisfies  $\delta \mathcal{J} = 0$  minimizes  $\mathcal{J}$  and provides the desired optimal control. Proceeding as in reference 3 the problem solution can be summarized as follows:

$$\underline{u}^* = W_3^{-1} B^T \underline{\lambda}(t) \quad (16)$$

where  $\underline{\lambda}$  satisfies the system of equations

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \hat{A} \begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} + \begin{Bmatrix} \underline{f} \\ \underline{0} \end{Bmatrix} \quad (17a)$$

$$\text{for } \hat{A} = \begin{bmatrix} A & BW_3^{-1}B^T \\ W_1 & -A^T \end{bmatrix} \quad (17b)$$

$$\underline{x}(0) = \underline{x}_0 \quad (18a)$$

$$\lim_{t \rightarrow \infty} \underline{x}(t) = \underline{0} \quad (18b)$$

$$\lim_{t \rightarrow \infty} \underline{f}(t) = \underline{0} \quad (18c)$$

If  $\underline{\lambda}$  can be found in terms of  $\underline{x}$  and  $\underline{f}$ , equation (16) will provide an expression for the optimal control.

The solution of the related homogeneous system

$$\begin{Bmatrix} \dot{\underline{x}} \\ \dot{\underline{\lambda}} \end{Bmatrix} = \hat{A} \begin{Bmatrix} \underline{x}_0 \\ \underline{\lambda}_0 \end{Bmatrix} \quad (19)$$

is

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} = e^{\hat{A}t} \begin{Bmatrix} \underline{x}_0 \\ \underline{\lambda}_0 \end{Bmatrix} \quad (20)$$

Assume for simplicity that the eigenvalues of  $\hat{A}$  are all distinct, so that the Jordan canonical form  $\hat{J}$  of  $\hat{A}$  is diagonal. Form a matrix  $X$  composed of the eigenvectors of  $\hat{A}$  (as its columns) such that

$$\hat{J} = X^{-1}\hat{A}X = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix} \quad (21)$$

has the negative eigenvalues  $\mu_i$  of  $\hat{A}$  as its diagonal elements, arranged such that  $\mu_1 < \mu_2 < \dots < \mu_n < 0$ .

Partition  $X$  and  $X^{-1}$  as  $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$  and  $\begin{bmatrix} X_{11}^{(-1)} & X_{12}^{(-1)} \\ X_{21}^{(-1)} & X_{22}^{(-1)} \end{bmatrix}$ , respectively. Then the solution to the homogeneous equation (19) is

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix}_h = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} e^{\Lambda t} & 0 \\ 0 & e^{-\Lambda t} \end{bmatrix} \begin{bmatrix} X_{11}^{(-1)} & X_{12}^{(-1)} \\ X_{21}^{(-1)} & X_{22}^{(-1)} \end{bmatrix} \begin{Bmatrix} \underline{x}_0 \\ \underline{\lambda}_0 \end{Bmatrix}, \quad (22)$$

expressible as

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix}_h = \begin{Bmatrix} X_{11}e^{\Lambda t}\tilde{\underline{c}}_1 + X_{12}e^{-\Lambda t}\tilde{\underline{c}}_2 \\ X_{21}e^{\Lambda t}\tilde{\underline{c}}_1 + X_{22}e^{-\Lambda t}\tilde{\underline{c}}_2 \end{Bmatrix} \quad (23a)$$

where

$$\tilde{\underline{c}}_1 = X_{11}^{(-1)}\underline{x}_0 + X_{12}^{(-1)}\underline{\lambda}_0 \quad (23b)$$

$$\tilde{\underline{c}}_2 = X_{21}^{(-1)} \underline{x}_0 + X_{22}^{(-1)} \underline{\lambda}_0 \quad (23c)$$

The variation of parameters method can now be used to find the general solution to the nonhomogeneous system

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} = \begin{Bmatrix} X_{11}e^{\Lambda t} \underline{c}_1 + X_{12}e^{-\Lambda t} \underline{c}_2 \\ X_{21}e^{\Lambda t} \underline{c}_1 + X_{22}e^{-\Lambda t} \underline{c}_2 \end{Bmatrix} \quad (24a)$$

where

$$\begin{Bmatrix} X_{11}e^{\Lambda t} \dot{\underline{c}}_1 + X_{12}e^{-\Lambda t} \dot{\underline{c}}_2 \\ X_{21}e^{\Lambda t} \dot{\underline{c}}_1 + X_{22}e^{-\Lambda t} \dot{\underline{c}}_2 \end{Bmatrix} = \begin{Bmatrix} \underline{f} \\ \underline{0} \end{Bmatrix} \quad (24b)$$

is to be solved and integrated to find  $\underline{c}_1$  and  $\underline{c}_2$ .

The solution to equation (24b) is

$$\begin{Bmatrix} \dot{\underline{c}}_1 \\ \dot{\underline{c}}_2 \end{Bmatrix} = \begin{bmatrix} e^{-\Lambda t} & 0 \\ 0 & e^{\Lambda t} \end{bmatrix} \begin{Bmatrix} X_{11}^{(-1)} \underline{f} \\ X_{21}^{(-1)} \underline{f} \end{Bmatrix} \quad (25a)$$

or

$$\begin{Bmatrix} \dot{\underline{c}}_1 \\ \dot{\underline{c}}_2 \end{Bmatrix} = \begin{Bmatrix} e^{-\mu_1 t} \underline{r}_1^T \underline{f} \\ \vdots \\ e^{-\mu_n t} \underline{r}_n^T \underline{f} \\ e^{\mu_1 t} \underline{s}_1^T \underline{f} \\ \vdots \\ e^{\mu_n t} \underline{s}_n^T \underline{f} \end{Bmatrix} \quad (25b)$$

where

$$X_{11}^{(-1)} = \begin{bmatrix} \underline{r}_1^T \\ \vdots \\ \underline{r}_n^T \end{bmatrix} \quad \text{and} \quad X_{21}^{(-1)} = \begin{bmatrix} \underline{s}_1^T \\ \vdots \\ \underline{s}_n^T \end{bmatrix} \quad (25c,d)$$

Integrating,

$$\underline{c}_1 = \underline{\gamma}_1 + \int e^{-\Lambda t} X_{11}^{(-1)} \underline{f} dt \quad (26a)$$

$$\underline{c}_2 = \underline{\gamma}_2 + \int e^{-\Lambda t} X_{21}^{(-1)} \underline{f} dt \quad (26b)$$

where the  $\gamma_{1i}$  and  $\gamma_{2i}$  ( $i = 1, \dots, n$ ) are  $2n$  integration constants and the integrals are indefinite integrals. Substituting into equation (24a) yields

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} = \begin{Bmatrix} X_{11} e^{\Lambda t} \left( \underline{\gamma}_1 + \int e^{-\Lambda t} X_{11}^{(-1)} \underline{f} dt \right) + X_{12} e^{-\Lambda t} \left( \underline{\gamma}_2 + \int e^{\Lambda t} X_{21}^{(-1)} \underline{f} dt \right) \\ X_{21} e^{\Lambda t} \left( \underline{\gamma}_1 + \int e^{-\Lambda t} X_{11}^{(-1)} \underline{f} dt \right) + X_{22} e^{-\Lambda t} \left( \underline{\gamma}_2 + \int e^{\Lambda t} X_{21}^{(-1)} \underline{f} dt \right) \end{Bmatrix} \quad (27)$$

Applying terminal conditions on  $\underline{x}(t)$  (eq. (18b)) leads to the result that  $\gamma_2 = \underline{0}$ , so that equation (27) simplifies to

$$\begin{Bmatrix} \underline{x} \\ \underline{\lambda} \end{Bmatrix} = \begin{Bmatrix} X_{11} \underline{\zeta} + X_{12} \underline{\Psi} \\ X_{21} \underline{\zeta} + X_{22} \underline{\Psi} \end{Bmatrix} \quad (28a)$$

where

$$\underline{\zeta} = e^{\Lambda t} \left( \gamma_1 + \int e^{-\Lambda t} X_{11}^{(-1)} \underline{f} dt \right) \quad (28b)$$

and

$$\underline{\Psi} = e^{-\Lambda t} \int e^{\Lambda t} X_{21}^{(-1)} \underline{f} dt \quad (28c)$$

If now we solve for  $\underline{\lambda}$  in terms of  $\underline{x}$ , we obtain the result

$$\underline{\lambda} = \left( X_{21} X_{11}^{-1} \right) \underline{x} + \left( X_{22} - X_{21} X_{11}^{-1} X_{12} \right) \underline{\Psi} \quad (29a)$$

or, equivalently,

$$\underline{\lambda} = \left( X_{21} X_{11}^{-1} \right) \underline{x} + X_{22}^{(-1)-1} e^{-\Lambda t} \int e^{\Lambda t} X_{21}^{(-1)} \underline{f} dt \quad (29b)$$

where  $X_{22}^{(-1)-1} = (X_{22} - X_{21} X_{11}^{-1} X_{12})^{-1}$  is the inverse of the lower right-hand partition of  $X^{-1}$ , as defined before. Applying equation (16) we have the result that the optimal control is

$$\underline{u}^* = \left( W_3^{-1} B^T X_{21} X_{11}^{-1} \right) \underline{x} + \left( W_3^{-1} B^T X_{22}^{(-1)-1} e^{-\Lambda t} \int e^{\Lambda t} X_{21}^{(-1)} \underline{f}(t) dt \right) \quad (30)$$

This is a simple form of the result obtained by Salukvadze in 1961 (ref. 1). If the disturbance  $\underline{f}$  is set equal to  $\underline{0}$ , the optimal control reduces to

$$\underline{u}^* = \left( W_3^{-1} B^T X_{21} X_{11}^{-1} \right) \underline{x} \quad (31)$$

By comparison with the solution to the LQR problem (ref. 4), it is apparent that

$$-X_{21} X_{11}^{-1} = P \quad (32)$$

where P is the solution to the Algebraic Riccati Equation (A.R.E.)

$$PA + A^T P - PBW_3^{-1}B^T P + W_1 = 0 \quad (33)$$

Making this substitution for  $-X_{21} X_{11}^{-1}$  into equation (30),

$$\underline{u}^* = -W_3^{-1}B^T P \underline{x} + W_3^{-1}B^T X_{22}^{(-1)-1} e^{-\Lambda t} \int e^{\Lambda t} X_{21}^{(-1)} \underline{f}(t) dt \quad (34)$$

If the indefinite integral in equation (30) is integrated repeatedly by parts, the control can be expressed as

$$\underline{u}^* = -W_3^{-1}B^T P \underline{x} - W_3^{-1}B^T X_{22}^{(-1)-1} \sum_{r=0}^{\infty} (-\Lambda^{-1})^{r+1} X_{21}^{(-1)} \underline{f}^{(r)}(t) \quad (35)$$

Assume, conservatively, that no derivatives of  $\underline{f}(t)$  are available. Then equation (35) reduces to

$$^{(0)}\underline{u}^* = -W_3^{-1}B^T P \underline{x} + W_3^{-1}B^T X_{22}^{(-1)-1} \Lambda^{-1} X_{21}^{(-1)} \underline{f} \quad (36)$$

where the "(0)" indicates that only the term for  $r=0$  has been retained and the "~" indicates that the resultant control only approximates the optimal. It can be shown readily that

$$X_{21}^{(-1)} = -X_{22}^{(-1)} X_{21} X_{11}^{-1} = X_{22}^{(-1)} P \quad (37)$$

so that

$$^{(0)}\underline{u}^* = -W_3^{-1}B^T P \underline{x} + W_3^{-1}B^T (X_{22}^{(-1)-1} \Lambda^{-1} X_{22}^{(-1)}) P \underline{f} \quad (38)$$

It can also be shown (not so readily) that

$$X_{22}^{(-1)-1} \Lambda^{-1} X_{22}^{(-1)} = \bar{A}^{-T} \quad (39)$$

where

$$\bar{A} = A - BW_3^{-1}B^T P \quad (40)$$

is the state matrix for the closed-loop system.

Substituting into equation (38)

$${}^{(0)}\underline{\hat{u}}^* = -\underline{W}_s^{-1}\underline{B}^T\underline{P}\underline{x} + \underline{W}_s^{-1}\underline{B}^T\bar{\underline{A}}^{-T}\underline{P}\underline{f} \quad (41)$$

By using equations (17b) and (21) one can show by simple matrix manipulations that

$$\bar{\underline{A}} = \underline{X}_{11}\underline{\Lambda}\underline{X}_{11}^{-1} \quad (42)$$

Equation (38) can then be expressed alternatively as

$${}^{(0)}\underline{\hat{u}}^* = -\underline{W}_s^{-1}\underline{B}^T\underline{P}\underline{x} + \underline{W}_s^{-1}\underline{B}^T(\underline{X}_{11}\underline{\Lambda}^{-1}\underline{X}_{11}^{-1})^T\underline{P}\underline{f} \quad (43)$$

or, applying equation (32), as

$${}^{(0)}\underline{\hat{u}}^* = -\underline{W}_s^{-1}\underline{B}^T\underline{P}\underline{x} + \underline{W}_s^{-1}\underline{B}^T(\underline{X}_{21}\underline{\Lambda}^{-1}\underline{X}_{11}^{-1})^T\underline{P}\underline{f} \quad (44)$$

## CONTROL EVALUATION

### Physical Realizability of the Control

The control (35) is physically realizable if the states and sufficient derivatives of  $\underline{f}(t)$  are accessible, if the coefficients are real, and if the higher order terms are negligible.

It can be shown by a state-transition-matrix approach that the coefficients are always real, whether or not the eigenvalues are real. Also, for eigenvalues of large enough modulus  $(-\underline{\Lambda}^{-1})^{r+1}$  rapidly converges to the zero matrix with increasing  $r$ , so that coefficients of higher order terms rapidly disappear as well. Given slowly varying disturbances the derivatives of  $\underline{f}(t)$  will be small, and the convergence of the higher order terms will be even more rapid.

### Transfer Function Matrix

Neglecting higher order terms the transfer function matrix for the closed-loop system is given by

$$\underline{X}(s) = (s\underline{I} - \bar{\underline{A}})^{-1}(\underline{I} - \underline{\mathfrak{H}}\underline{F}(s)) \quad (45a)$$

where

$$\mathcal{H} = -BW_3^{-1}B^T[X_{22}^{(-1)-1}\Lambda^{-1}X_{22}^{(-1)}]P \quad (46a)$$

$$= -BW_3^{-1}B^T[X_{11}\Lambda^{-1}X_{11}^{-1}]^TP \quad (46b)$$

$$= -BW_3^{-1}(\bar{A}^{-1}B)^TP \quad (46c)$$

### Control Stability and Stability Robustness

Since the control feedback gains are simply LQR feedback gains the closed loop system is stable and enjoys the stability robustness characteristics guaranteed by LQR theory,<sup>1</sup> viz., a minimum of 60° phase margin, infinite positive gain margin, and 6 dB negative gain margin.

### SAMPLE PROBLEMS

#### First Order Problem

Let a first-order problem be described as follows:

Find  $u = u^*$  which minimizes  $J$  where

$$J = \frac{1}{2} \int_0^{\infty} (w_1 x^2 + w_3 u^2) dt \quad (47a)$$

for the system

$$\dot{x} = ax + bu + f \quad (47b)$$

with initial condition

$$x(0) = x_0 \quad (47c)$$

and terminal conditions

---

<sup>1</sup>Phase margin and gain margin in the multiple-input multiple-output case are measured by diagonal perturbation; i.e., the same change in each channel (ref. 5).



$$\lim_{t \rightarrow \infty} x(t) = 0 \quad (47d)$$

$$\lim_{t \rightarrow \infty} f(t) = 0 \quad (47e)$$

The Hamiltonian matrix is  $\begin{bmatrix} a & \frac{b^2}{w_3} \\ w_1 & -a \end{bmatrix}$ , for which the eigenvalue matrix and associated eigenvector matrix are

$$\begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -\mu \end{bmatrix} \quad (48a)$$

where

$$\mu = - \sqrt{a^2 + b^2 \frac{w_1}{w_3}} \quad (48b)$$

and

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} b^2 & -b^2 \\ w_3(\mu - a) & w_3(\mu + a) \end{bmatrix} \quad (48c)$$

Inverting,

$$X^{-1} = \begin{bmatrix} X_{11}^{(-1)} & X_{12}^{(-1)} \\ X_{21}^{(-1)} & X_{22}^{(-1)} \end{bmatrix} = \begin{bmatrix} \frac{\mu + a}{2b^2\mu} & \frac{1}{2w_3\mu} \\ \frac{-(\mu - a)}{2b^2\mu} & \frac{1}{2w_3\mu} \end{bmatrix} \quad (48d)$$

Equations (36), (38), and (41) each yield the result

$${}^{(0)}\hat{u}^* = \left( \frac{\mu - a}{b} \right) x - \left( \frac{\mu - a}{b\mu} \right) f, \quad (49)$$

for which the closed-loop transfer function is given by

$$X(s) = \frac{\alpha}{\mu(s - \mu)} F(s) \quad (50)$$

With LQR F/B alone the control is

$$u_L^* = \left( \frac{\mu - a}{b} \right) x, \quad (51)$$

for which the closed loop transfer function is given by

$$X(s) = \frac{1}{s - \mu} F(s) \quad (52)$$

Note from (48b) that the modulus of  $\mu$  is always greater than that of  $a$ , so that inclusion of the feedforward control term always result in better disturbance rejection than that from control with LQR F/B alone.

Comparison of the performance indices  $J_L^*$  and  ${}^{(0)}J^*$  corresponding to  $u_L^*$  and  ${}^{(0)}\hat{u}$ , respectively, is very messy algebraically. However it can be shown, for  $x_0 = 0$  and arbitrary dwindling function  $f(t)$ , that  ${}^{(0)}J^* < J_L^*$  if  $a < 0$  (i.e., if the system is open-loop stable), and that  ${}^{(0)}J^* < J_L^*$  if  $a > 0$  if  $-\mu > 4/3 a$  (i.e., if  $w_1/w_3 > 7a^2/9b^2$ ). For  $x_0 \neq 0$  and  $f \equiv 0$  the two controls  $u_L^*$  and  ${}^{(0)}\hat{u}$  are identical, as are their respective performance indices.

## Second Order Problem

Suppose now that the solution method is applied to the space-experiment disturbance-rejection problem raised previously in this paper. Consider the first mathematical model of the system, so that the problem is as follows:

Find  $u = u^*$  which minimizes  $J$  where

$$J = \frac{1}{2} \int_0^\infty (\underline{w}^T W_1 \underline{x} + w_3 u^2) dt \quad (53a)$$

with  $W_1 = \begin{bmatrix} w_{11} & 0 \\ 0 & w_{22} \end{bmatrix}$  for the system

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{\alpha}{m} \end{Bmatrix} u + \begin{Bmatrix} 0 \\ \frac{k}{m} d(t) + \frac{c}{m} \dot{d}(t) \end{Bmatrix} \quad (53b)$$

given the conditions

$$\underline{x}(0) = \underline{x}_0 \quad (53c)$$

$$\lim_{t \rightarrow \infty} \underline{x}(t) = \underline{0} \quad (53d)$$

$$\lim_{t \rightarrow \infty} \underline{f}(t) = \underline{0} \quad (53e)$$

Let the system parameters be  $m = 100$  lbm,  $k = 0.3$  lbf/ft, and  $c = 0.000622$  lbf-sec/ft, corresponding to damping ratio  $\zeta = 0.1$  percent.

Selected results are found to be as previously reported in reference 3, where this particular problem was treated in detail. There it was noted that the performance index weights low frequency accelerations

by a factor proportional to  $\left( \frac{w_{11}}{\omega^4} + \frac{w_{22}}{\omega^2} \right)^{1/2}$ , so that low frequencies are more heavily weighted, as

desired. The result is that for low frequencies the inclusion of feedforward in the control can lead to acceleration reductions orders of magnitude below that afforded by simple LQR feedback alone, without leading to poor performance at higher frequencies. Different frequency-weighting schemes, of course, would lead to different disturbance-rejection characteristics (ref. 6).

For any given set of weights  $w_{11}$ ,  $w_{22}$ , and  $w_3$ , the closed loop gain for the system with LQR feedback alone can be compared with that for the system with LQR feedback plus proportional feedforward, at a given frequency. Table I presents the factors by which the DC level ( $\omega = 0$ ) for the closed-loop system is reduced by merely adding a proportional feedforward term to standard LQR feedback. This represents, in general, the factor by which the closed-loop Bode- $\alpha$  plot for a given set of weights is lowered at low frequencies by the addition of proportional feedforward. Note that for all combinations of weights considered—each weight being allowed to vary over four orders of magnitude—the reduction factor lies between zero and one, varying from  $6.00 \times 10^{-4}$  to  $9.86 \times 10^{-1}$ . The optimization procedure accomplishes this task by moving the transfer function zeros to reduce the gains in accordance with the frequency-, state-, and control-weightings present in the performance index. The poles of the transfer function depend on the feedback gains alone, and do not vary with feedforward changes.

## CONCLUSION

This paper has developed a general expression for an optimal control in the case of system state equations with dwindling forcing (disturbance) terms included, given a quadratic performance index. The control, when expanded in series form, has been found to entail constant, real-valued feedback gains identical to those determined by the standard LQR approach, along with constant, real-valued feedforward gains premultiplying the disturbance terms and their derivatives.

It has been found that the control offers significant disturbance-rejection improvements over a control that uses LQR feedback alone, without sacrificing robustness. In at least the 1st order system, sufficient conditions have been presented for the simplest F/B plus F/F control which result in a lower performance index than with LQR F/B alone. With large enough closed-loop system eigenvalues and slowly varying disturbances, the conclusion was made that only a few feedforward terms are needed to approximate closely the actual optimal controls.

Five mathematical models of a one-dimensional disturbance-rejection problem were suggested, each of which is in a form amenable to the optimal control approach presented by this paper. Application was made of the optimal control method to one of these models, leading to the same numerical values of gains (both F/B and F/F) as found previously using a nonlinear-algebra formulation (ref. 3).

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TABLE I. – DC GAIN REDUCTION FACTORS WITH THE ADDITION OF PROPORTIONAL  
FEEDFORWARD, FOR A SINGLE-MASS SYSTEM WITH  $M=100$  LBM,  
 $K=0.3$  LBF/FT, AND  $C=0.000622$  LBF-SEC/FT ( $\zeta=0.1\%$ )

$W_3=0.01$

$\begin{matrix} W_{22} \\ W_{11} \end{matrix}$	0.01	0.1	1	10	10
0.01	←		5.989229 E-2		→
0.1	←		1.897025 E-2		→
1	←		5.999892 E-3		→
10	←		1.897363 E-3		→
100	←		5.999999 E-4		→

$W_3=0.1$

$\begin{matrix} W_{22} \\ W_{11} \end{matrix}$	0.01	0.1	1	10	10
0.01	←		1.864109 E-1		→
0.1	←		5.989229 E-2		→
1	←		1.897025 E-2		→
10	←		5.999892 E-3		→
100	←		1.897363 E-3		→

$W_3=1$

$\begin{matrix} W_{22} \\ W_{11} \end{matrix}$	0.01	0.1	1	10	10
0.01	←		5.144958 E-1		→
0.1	←		1.864109 E-1		→
1	←		5.989229 E-2		→
10	←		1.897025 E-2		→
100	←		5.999892 E-3		→

$W_3=10$

$\begin{matrix} W_{22} \\ W_{11} \end{matrix}$	0.01	0.1	1	10	10
0.01	←		8.846517 E-1		→
0.1	←		5.144958 E-1		→
1	←		1.864109 E-1		→
10	←		5.989229 E-2		→
100	←		1.897025 E-2		→

$W_3=100$

$\begin{matrix} W_{22} \\ W_{11} \end{matrix}$	0.01	0.1	1	10	10
0.01	←		9.863939 E-1		→
0.1	←		8.846517 E-1		→
1	←		5.144958 E-1		→
10	←		1.864109 E-1		→
100	←		5.989229 E-2		→

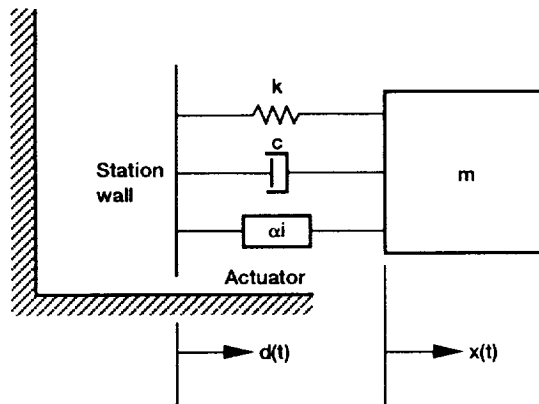


Figure 1.—Physical representation of modeled single degree-of-freedom system.

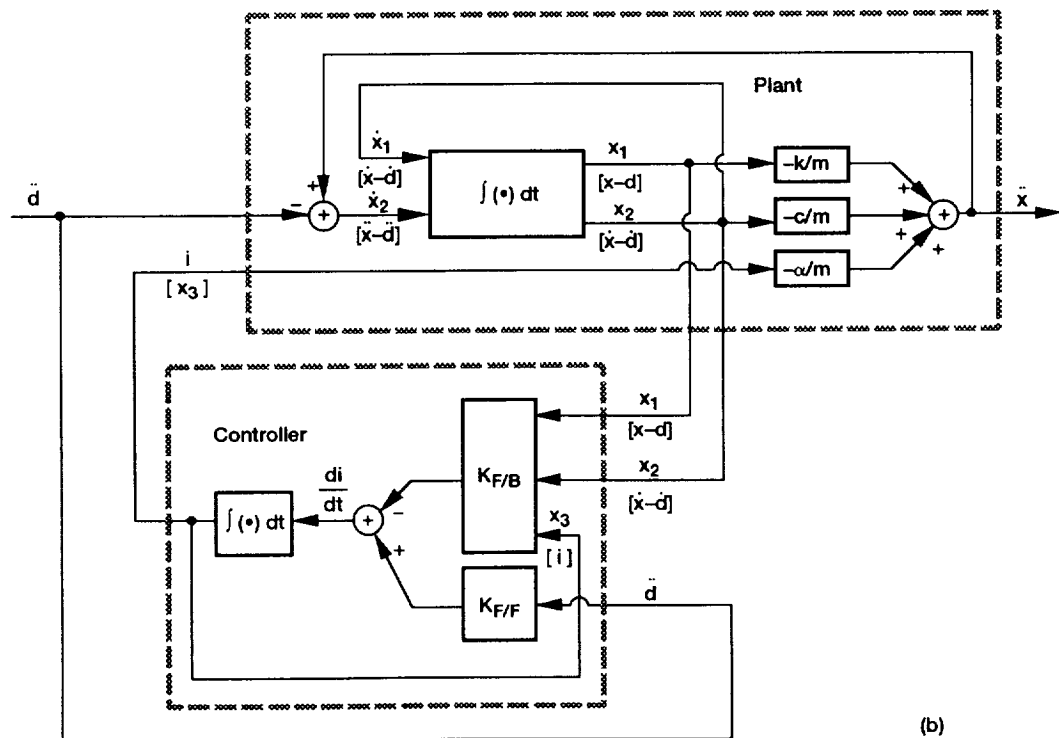
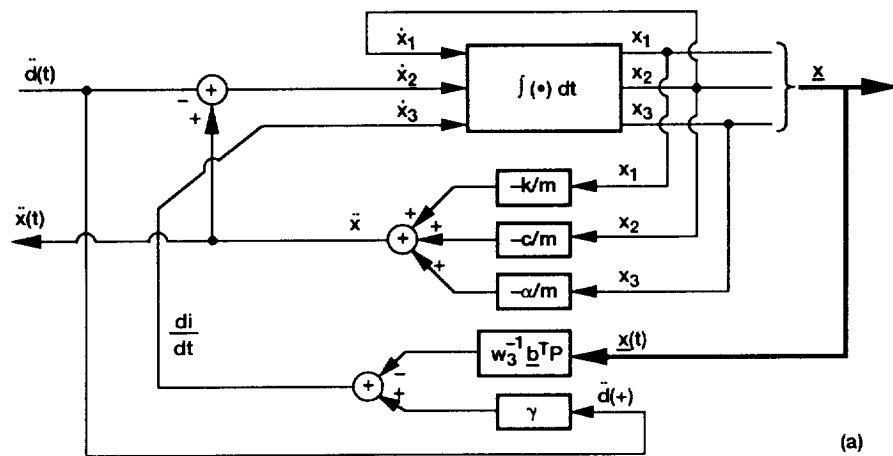


Figure 2.—Controller Block diagrams for constant feedforward gain. Both figures (a) and (b) are equivalent implementations of the constant feedforward gain controller.

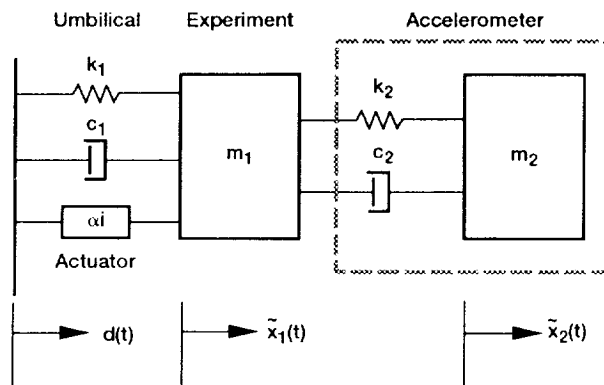


Figure 3.—Physical representations of modeled systems using two masses  $m_1$  and  $m_2$ , representing the experimental mass and an attached accelerometer.

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